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FLOWS FOR DIFFERENTIABLE VECTOR FIELDS ON CONJUGATE BANACH SPACES

RICHARD A. GRAFF

Introduction

In [6] the author showed that many of the manifolds of maps which are used in global nonlinear analysis and previously had been regarded solely as infinite-dimensional Banach manifolds do in fact possess the much richer structure of objects which the author called Bw^* manifolds. The author introduced an approach to a theory of differential calculus for maps between Bw^* spaces as a tool for the development of differentiable Bw^* manifold theory, and isolated two classes of differentiable maps which the author labeled C^k and $\mathfrak{U}^k$. These classes of maps were shown to be closed under composition, and the inverse function theorem was established for $\mathfrak{U}^k$ maps.

The main result of this paper is Theorem 6.3, which will complete the development of differential calculus for $\mathfrak{U}^k$ maps by showing that $\mathfrak{U}^k$ vector fields generate $\mathfrak{U}^k$ flows. Since a Bw^* space can also be regarded as a Banach space, and since the $\mathfrak{U}^k$ vector fields represent a subclass of the usual C^k vector fields under this identification, this existence theorem for flows generated by $\mathfrak{U}^k$ vector fields is actually a strengthened version of the existence theorem for flows generated by C^k vector fields on Banach spaces. In fact, the ordinary existence theorem will be used in the proof of this new version in a manner reminiscent of the use of the ordinary inverse function in the proof of the inverse function theorem for $\mathfrak{U}^k$ maps.

Since a detailed account of the relevant basic theory of Bw^* spaces and differential calculus map be found in [6], the reader will be assumed to be acquainted with the material covered in that reference from the beginning of Chapter 2 through the first half of Chapter 6. However, for the sake of self-sufficiency, a brief review of essential information is included in §1.

1. Bw^* Spaces and Differential Calculus

Notation and terminology. Let X be a bw^* space. Then X' is a Banach space as is X'', and the natural linear injection of X into X'' is onto. Furthermore,

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the topology induced on X'' by its identification with X is identical to the bounded weak-star topology induced on X'' by its duality with X'. Conversely, every conjugate Banach space reequipped with the bounded weak-star topology is a bw^* space, so it will be convenient to avoid an abstract definition in this paper and simply think of a bw^* space in this somewhat more concrete fashion. The letter X will be used to denote a bw^* space, and the letter E will be used instead of X'' to denote the second dual of X. Note that E can be thought of as a retopologized version of X. Since the topology on E can be recovered by taking the second dual of X, bw^* spaces can be thought of as possessing two topologies. E will be assumed to possess a norm in which the closed unit ball is compact in X. $B_0(1)$ will denote the closed unit ball with the topology which it inherits from E, and K will denote this ball with the topology which it inherits from X. It is shown in [6, Chapter 2] that $X = \lim_{n \in N} nK$. $L^r(X, X)$ will denote the space of continuous r-linear maps from X^r to X with the topology of uniform convergence on bounded sets. If X' is separable, i.e., if K is metrizable, then X will be called a Bw^* space.

If X_1 and X_2 are bw^* spaces, W open in X_1, and f: $W \to X_2$, then f may also be regarded as a map from an open subset of E_1 to E_2. If f is C^k between E_1 and E_2, it will be said to be strongly C^k; if f is C^k between X_1 and X_2, it will simply be said to be C^k (or weakly C^k, if there is need to emphasize that the weaker topology is the one involved). The same conventions hold for f continuous (or weakly continuous) versus strongly continuous. If f is weakly C^k, $k \geqslant 1$, then f is strongly C^{k-1}, and $D^{k-1}f$ is locally Lipschitz between the normed spaces.

Let X_1, X_2, X_3 be bw^* spaces, W open in X_1, and g: $W \to L(X_2, X_3)$ a continuous map. We will say g is $\mathfrak{U}$-regulated if g satisfies the following condition: for each compact subset C of W there exist continuous semi-norms λ on X_1 and ν on X_2 such that, for each $x, y \in W$ and $z \in X_2$, $\|(g(y) - g(x))(z)\| \leqslant \lambda(y - x)\|z\| + \|y - x\|\nu(z)$, where $\|v\|$ denotes the norm of v in E_i for each $v \in X_i$. Note that, for the case $X_1 = X_2$, we may replace both λ and ν in this definition by a single semi-norm (for instance, $\lambda + \nu$). We will call f: Wx_2 a $\mathfrak{U}^1$ map if f is C^1 and Df is $\mathfrak{U}$-regulated.

The concept of a $\mathfrak{U}^k$ map for $k > 1$ is defined inductively as follows: let $k \geqslant 2$, and assume that the definition of a $\mathfrak{U}^{k-1}$ map has already been introduced. Note that to each C^1 map: f: $W \to X_2$ we may associate a map Tf: $W \times X_1 \to X_2 \times X_2$ which is defined by $Tf(x, y) = (f(x), Df(x)(y))$. We will call f a $\mathfrak{U}^k$ map if f is C^1 and Tf is $\mathfrak{U}^{k-1}$. We will call f a $\mathfrak{U}^\infty$ map if f is $\mathfrak{U}^k$ for each $k \in N$. If f is $\mathfrak{U}^k$ for $k \geqslant 1$, then it follows that f is both C^k and strongly C^k. If X_1 is finite-dimensional, then f is $\mathfrak{U}^k \Leftrightarrow f$ is strongly C^k and D^kf is a locally Lipschitz map between the associated Banach spaces.

The coordinate transformations between charts belonging to the naturally-arising atlases on many familiar manifolds of maps (e.g., spaces of maps of Sobolev, Lipschitz, and Hölder type) are $\mathcal{U}^\infty$, as are many of the maps induced between such function spaces by smooth nonlinear partial differential operators.

2. Integral curves

Let W be open in X, and $f\colon W \to X$ a vector field which is both continuous and strongly continuous. An integral curve α for f with initial condition $x \in W$ is a C^1 map from an interval (a, b) to X such that $a < 0 < b$, $\alpha((a, b)) \subset W$, $\alpha(0) = x$, and $\alpha'(t) = f(\alpha(t))$ for each $t \in (a, b)$, where the derivative is of course taken in X. A *strong integral curve* for f is an integral curve for f in the usual, Banach space sense. It is obvious that a strong integral curve for f is also an integral curve.

2.1. Lemma. *An integral curve for f is also a strong integral curve for f.*

Proof. Let $\alpha\colon (a, b) \to X$ be an integral curve for f, and let $\varepsilon > 0$. Then $\alpha([a + \varepsilon, b - \varepsilon])$ is compact in X, and so there exists $c > 0$ such that $\|f(\alpha(t))\| \leqslant c$ for each $t \in [a + \varepsilon, b - \varepsilon]$. It follows that, for each $r, t \in [a + \varepsilon, b - \varepsilon]$,

$$\|\alpha(t) - \alpha(s)\| = \left\| \int_r^t f \circ \alpha(s)\, ds \right\| \leqslant \int_r^t \|f \circ \alpha(s)\|\, ds \leqslant c|t - r|.$$

Thus α is strongly continuous on (a, b). Since $\alpha' = f \circ \alpha$ is strongly continuous on (a, b), it is immediate that α is strongly C^1 on (a, b), and hence that α is a strong integral curve for f. q.e.d.

Since integral curves for f coincide with strong integral curves for f in the case where f is both continuous and strongly continuous (which is the only kind of vector field to be considered in this paper), the distinction between these two definitions will be dropped, and the term "integral curve" will be used to refer to a curve which satisfies either definition.

2.2. Lemma. *Let W be open in S, and $f\colon W \to X$ a C^1 vector field. Then an integral curve for f exists through each point of W, and maximal integral curves are unique.*

Proof. Since f is C^1, it is locally uniformly Lipschitz in E. Thus the result is simply the standard existence and uniqueness theorem for integral curves for a locally uniformly Lipschitz vector field in a Banach space.

3. The Cauchy problem for continuous families of bounded linear operators

Let Z be a topological space, $a < b$, V a sequentially complete locally convex space, and $g\colon Z \times [a, b] \times [a, b] \to V$ a continuous map. Define $h\colon Z \times [a, b] \times [a, b] \to V$ by $h(z, t, r) = \int_r^t g(z, s, r)\, ds$. Then it is immediate that h is continuous.

Now assume that Z is compact and that $A\colon Z \times [a, b] \to L(X, X)$ is continuous. Define a sequence $\{U_n\}_{n \geqslant 0}$ of maps from $Z \times [a, b] \times [a, b]$ to $L(X, X)$ as follows: $U_0(z, t, r) = \mathrm{Id}_X$ for each $(z, t, r) \in Z \times [a, b] \times [a, b]$, and $U_n(z, t, r) = \int_r^t A(z, s) U_{n-1}(z, s, r)\, ds$ for each t, r. If $U_{n-1}(\cdot,\cdot,\cdot)$ is continuous then, since the topology on $L(X, X)$ is the compact-open topology and since X is a k-space, it follows that the map $(z, t, r) \to A(z, t) U_{n-1}(z, t, r)$ is continuous. Thus, by the preceding paragraph, $U_n(\cdot,\cdot,\cdot)$ is continuous.

To see that $\sum_{n=0}^{\infty} U_n(\cdot,\cdot,\cdot)$ exists note that, since $Z \times [a, b] \times [a, b]$ is compact, $A(\cdot,\cdot)$ is bounded in $L(X, X)$ and hence also in $L(E, E)$. So there exists $c > 0$ such that $\|A(\cdot,\cdot)\| \leqslant c$. It follows by a standard argument that

$$\| U_n(z, t, r) \| \leqslant \frac{c^n |t - r|^n}{n!} \leqslant \frac{c^n |a - b|^n}{n!}$$

for each $n \geqslant 0$, and hence that the partial sums $\{s_n = \sum_{j=0}^{n} U_j\}_{n \geqslant 0}$ are uniformly Cauchy in $L(E, E)$. Since the uniformity on $L(X, X)$ is weaker than the uniformity which $L(X, X)$ inherits as a subspace of $L(E, E)$, it follows that the sequence $\{s_n\}$ is a uniformly Cauchy sequence of continuous maps into $L(X, X)$. Since $L(X, X)$ is complete, we conclude that $U(\cdot,\cdot,\cdot) = \sum_{n=0}^{\infty} U_n(\cdot,\cdot,\cdot)$ exists and is continuous, and that $U(z, t, r) = \int_r^t A(z, s) U(z, s, r)\, ds$ for each $z \in Z$, $t, r \in [a, b]$. This implies that, for each $x \in X$, $U(z, \cdot, r)(x)$ is a solution to the equation $du/dt = A(z, t)u(t)$ on $[a, b]$ with $u(r) = x$. Uniqueness of solutions to this equation also follows by a standard argument: it is easy to verify that $(\partial/\partial r) U(z, t, r)$ exists and equals $-U(z, t, r)A(z, r)$ for each $z \in Z$ and $t, r \in [a, b]$. It is easily seen by a similar argument that, if $v(\cdot)$ is any solution to the equation $du/dt = A(z, t)u(t)$ on $[a, b]$, then $(\partial/\partial r)(U(z, s, r)v(r)) = U(z, s, r)v'(r) - U(z, s, r)A(z, r)v(r) = 0$ for each $s \in [r, t]$, and hence that $v(\cdot) = U(z, \cdot, r)v(r)$ for each $r \in [a, b]$. Letting $v(s) = U(z, s, r)(x)$, the uniqueness of solutions implies that $U(z, t, s) U(z, s, r) = U(z, t, r)$ for each $t, s, r \in [a, b]$, and this composition is easily seen to be jointly continuous in all variables.

Incidentally, the standard argument shows that each $U(z, t, r)$ is invertible, and that $U(z, t, r)^{-1} = U(z, r, t)$. For,

$$\frac{\partial}{\partial s}(U(z, t, s) U(z, s, t)) = U(z, t, s)(A(z, s) - A(z, s)) U(z, s, t) = 0$$

for each $s \in [r, t]$, which implies that $U(z, t, r)U(z, r, t) = \mathrm{Id}_X$. Reversing the roles of r and t, the result follows.

The above construction extends immediately to the case where Z is assumed to be a k-space. The construction and uniqueness of $U(z,\cdot,\cdot)$ are apparent for each $z \in Z$, so it is only necessary to check the continuity of $U(\cdot,\cdot,\cdot)$. But since the product of a k-space with a compact space is again a k-space, it is sufficient to check the continuity of $U(\cdot,\cdot,\cdot)$ on $C \times [a, b] \times [a, b]$ for each compact subset C of Z, which we have already done.

4. Differentiability of continuous flows

4.1. Definition. Let f: $W \to X$ be a vector field which is both continuous and strongly continuous. A continuous map $F(\cdot,\cdot)$ from an open subset V of $\mathbf{R} \times E$ to E will be called a strong flow generated by f (i.e., a flow in the conventional sense) if $\{0\} \times W \subset V$ and if each $F(\cdot,x)$ is an integral curve for f. A strong flow F will be called a flow generated by f if V is open in $\mathbf{R} \times X$ and $F(\cdot,\cdot)$ is weakly continuous from V to X.

4.2. Theorem. *Let W be open in X, f: $W \to X$ a vector field which is C^1, and $F(\cdot,\cdot)$ a flow generated by f. Then F is C^1.*

Proof. It suffices to verify that F has continuous partial derivatives with respect to the time and space variables. By definition of flow, D_1F: $V \to L(\mathbf{R}, E) = E$ exists and is given by $D_1F = f \circ F$, which implies that D_1F is continuous.

To see that $D_2F(\cdot,\cdot)$ exists and is continuous on V, we proceed as follows: define $A(\cdot,\cdot)$: $V \to L(X, X)$ by $A(z, s) = Df(F(s, z))$ for each $(s, z) \in V$, and for each $z \in W$ let $U(z,\cdot,\cdot)$ be the (unique) evolution system generated by $A(z,\cdot)$. Let $(t, x) \in V$: choose $a, b \in \mathbf{R}$ and an open neighborhood Z of x such that $a < 0 < b$, $t \in (a, b)$, and $[a, b] \times Z \subset V$. By the results of §3, $U(\cdot,\cdot,\cdot)$ is continuous from $Z \times [a, b] \times [a, b]$ to $L(X, X)$, so $U(\cdot,\cdot, 0)$ is continuous on a neighborhood of (t, x) in V.

To see that $D_2F(\cdot,\cdot)$ exists at (t, x) and equals $U(x, t, 0)$, define G: $Z \times [a, b] \to L(X, X)$ by $G(y, s) = \int_0^1 Df(rF(s, y) + (1 - r)F(s, x))\, dr$ (by shrinking Z if necessary, we can ensure that the line segment from $F(s, y)$ to $F(s, x)$ is in W for each $y \in Z$ and $s \in [a, b]$). Let $U'(\cdot,\cdot,\cdot)$: $Z \times [a, b] \times [a, b] \to L(X, X)$ be the parametrized evolution system generated by $G(\cdot,\cdot)$. Again by the results of §3, $U'(\cdot,\cdot,\cdot)$ is continuous.

For each $y \in Z$, define $\alpha_y(\cdot)$ on $[a, b]$ by $\alpha_y(\cdot) = F(\cdot, y) - F(\cdot, x)$. Then, for each $s \in [a, b]$,

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(U'(y, t, s)\alpha_y(s)\right)\\
&\quad = U'(y, t, s)\alpha_y'(s) - U'(y, t, s)G(y, s)\alpha_y(s)\\
&\quad = U'(y, t, s)\Big[f(F(s, y)) - f(F(s, x))\\
&\qquad - \int_0^1 Df(rF(s, y) + (1 - r)F(s, x))(F(s, y) - F(s, x))\, dr\Big]\\
&\quad = 0
\end{aligned}$$

by the Fundamental Theorem of Calculus. Integrating this equation from 0 to t, we get $F(t, y) - F(t, x) = U'(y, t, 0)(y - x)$. Thus

$$\begin{aligned}
F(t, y) - F(t, x) - U(x, t, 0)(y - x)&\\
&= (U'(y, t, 0) - U(x, t, 0))(y - x)\\
&= (U'(y, t, 0) - U'(x, t, 0))(y - x).
\end{aligned}$$

Let λ be any continuous semi-norm on X, let p_λ: $X \to X_\lambda$ be the canonical projection, and define g: $Z \to L(X, X_\lambda)$ by $g(y) = p_\lambda \circ (U'(y, t, 0) - U'(x, t, 0))$. Since X is a D-space, there exist an open neighborhood Y of x in Z and a continuous semi-norm ν on X such that $g(Y) \subset L_\nu(X, X_\lambda)$ and g: $Y \to L_\nu(X, X_\lambda)$ is continuous. Since $g(x) = 0$, for each $\varepsilon > 0$ there exists a neighborhood Y_ε of x in Y such that $\| g(y)\|_{\nu,\lambda} \leqslant \varepsilon$ for every $y \in Y_\varepsilon$, where $\| \ \|_{\nu,\lambda}$ is the norm on $L_\nu(X, X_\lambda)$. Thus, for $y \in Y_\varepsilon$,

$$\begin{aligned}
\lambda(F(t, y) - F(t, x) - U(x, t, 0)(y - x))&\\
&= \lambda(g(y)(y - x))\\
&\leqslant \| g(y)\|_{\nu,\lambda}\nu(y - x) \leqslant \varepsilon\nu(y - x),
\end{aligned}$$

which implies that $D_2F(\cdot,\cdot)$ exists at (t, x) and equals $U(x, t, 0)$. Thus we have existence of D_2F at each point of V, and continuity of D_2F at each point V by the second paragraph of this proof.

4.3. Corollary. *Let W be open in X, f: $W \to X$ a $\mathfrak{A}^1$ vector field, and $F(\cdot,\cdot)$: $V \to X$ a flow generated by f. Then $F(\cdot,\cdot)$ is a $\mathfrak{A}^1$ map.*

Proof. Since a $\mathfrak{A}^1$ vector field is C^1, it follows from Theorem 4.2 that F is C^1. It merely remains for us to show that DF is $\mathfrak{A}$-regulated.

As in the proof of the above theorem, it is convenient to examine the partial derivatives D_1F and D_2F separately. We know that D_1F: $V \to X = L(\mathbf{R}, X)$ can be factored as $D_1F = f \circ F$, which implies that D_1F is C^1. By [6, Example 5.18], it follows that D_1F is $\mathfrak{A}$-regulated on V and hence also on any open subset of V.

We will examine D_2F on open subsets of V which are of particularly nice form. Assume that $(t, x) \in V$; let $a, b \in \mathbf{R}$ and Z open in X be as in the proof of Theorem 4.1, and let $A(\cdot,\cdot)$, and $U(\cdot,\cdot,\cdot)$ also be as in the proof of Theorem 4.1. Define $V_1 = (a, b) \times Z$. Note that, since $A(z, s) = Df(F(s, z))$ for each $(s, z) \subset V$, [6, Lemma 5.21] implies that $A(\cdot,\cdot)$ is $\mathcal{U}$-regulated on V. Let C be a compact subset of Z. Since $A(\cdot,\cdot)$ and $U(\cdot,\cdot, 0)$ are continuous on $[a, b] \times Z$, it follows that there is a constant $k > 0$ such that $\|A(\cdot,\cdot)\| \leqslant k$ and $\|U(\cdot,\cdot, 0)\| \leqslant k$ on $[a, b] \times C$. Let $z, w \in C$, $s \in [a, b]$, $y \in X$. Since $U(z, 0, 0) = U(w, 0, 0) = \mathrm{Id}_X$, the Fundamental Theorem implies that

$$\begin{aligned}
&\|U(z, s, 0)(y) - U(w, s, 0)(y)\| \\
&\quad = \left\| \int_0^s (A(z, r)U(z, r, 0)(y) - A(w, r)U(w, r, 0)(y))\, dr \right\| \\
&\quad \leqslant \left\| \int_0^s (A(z, r) - A(w, r))U(z, r, 0)(y)\, dr \right\| \\
&\qquad + \left\| \int_0^s A(w, r)(U(z, r, 0)(y) - U(w, r, 0)(y))\, dr \right\| \\
&\quad \leqslant \int_a^b \|(A(z, r) - A(w, r))U(z, r, 0)(y)\|\, dr \\
&\qquad + k\int_0^s \|U(z, r, 0)(y) - U(w, r, 0)(y)\|\, dr.
\end{aligned}$$

Applying Gronwall's inequality, we conclude that

$$\begin{aligned}
&\|U(z, s, 0)(y) - U(w, s, 0)(y)\| \\
&\qquad \leqslant e^{k(b-a)} \int_a^b \|(A(z, r) - A(w, r))U(z, r, 0)(y)\|\, dr.
\end{aligned}$$

Since $A(\cdot,\cdot)$ is $\mathcal{U}$-regulated on V, and $[a, b] \times C$ is a compact subset of V, there exists a semi-norm λ on X such that

$$\|(A(z, r) - A(w, r))(v)\| \leqslant \|z - w\|\lambda(v) + \lambda(z - w)\|v\|$$

for each $z, w \in C$, $r \in [a, b]$, and $v \in X$. By [6, Lemma 5.7] there exists a semi-norm ν on X such that $\lambda(U(z, r, 0)(y)) \leqslant \nu(y)$ for every $z \in C$, $r \in [a, b]$, and $y \in X$. Thus we obtain that

$$\begin{aligned}
&\|U(z, s, 0)(y) - U(w, s, 0)(y)\| \\
&\qquad \leqslant (b - a)e^{k(b-a)}(\|z - w\|\nu(y) + k\lambda(z - w)\|y\|).
\end{aligned}$$

Finally, for each $z, w \in C$, $s, s' \in [a, b]$, and $y \in X$,

$$\begin{aligned}
&\|D_2F(s', z)(y) - D_2F(s, w)(y)\| \\
&\quad = \|U(z, s', 0)(y) - U(w, s, 0)(y)\| \\
&\quad \leqslant \|U(z, s', 0)(y) - U(z, s, 0)(y)\| + \|U(z, s, 0)(y) - U(w, s, 0)(y)\| \\
&\quad = \left\|\int_s^{s'} A(z, r)U(z, r, 0)(y)\, dr\right\| + \|U(z, s, 0)(y) - U(w, s, 0)(y)\| \\
&\quad \leqslant k^2|s' - s| \cdot \|y\| + (b - a)e^{k(b-a)}(\|z - w\|\nu(y) + k\lambda(z - w)\|y\|).
\end{aligned}$$

The above inequality implies that D_2F is $\mathfrak{U}$-regulated on V_1. Since $DF = D_1F + D_2F$, and both D_1F and D_2F are $\mathfrak{U}$-regulated on V_1, it follows that DF is $\mathfrak{U}$-regulated on V_1.

Finally, since for each point $(t, x) \in V$ there exists a neighborhood V_1 of (t, x) in V such that the restriction of F to V_1 is $\mathfrak{U}^1$, from [6, Prop. 5.23] we see that F is $\mathfrak{U}^1$ on V.

5. Higher-order differentiability of continuous flows

We will next see that a continuous flow for a vector field is as highly differentiable as is the vector field. The proof is an adaptation of a standard induction argument. We will need to recall the notion of the tangent map associated to a differentiable map, and we will need a partial converse to [6, Proposition 4.14]. While we only need this converse for the case of vector fields on bw^* spaces, the proof yields a result of much greater generality, and so the more general result appears in the statement of Theorem 5.2. For the definitions of exponential and Schwartz spaces, refer to [6, Chapter 3]. If you do not want to bother with the proof of Theorem 5.2 in this generality, simply assume in the statement of Theorem 5.2 that Y is a bw^* space.

5.1. Definition. Let Y and Z be locally convex spaces, V open in Y, and $f\colon V \to Z$ a C^1 map. Then the map $Tf\colon V \times Y \to Z \times Z$ is defined by $Tf(x, y) = (f(x), Df(x)(y))$.

5.2. Theorem. *Let Y be a complete exponential Schwartz space, Z a complete locally convex space, V open in Y, $f\colon V \to Z$ a C^1 map, and $k \in N$. Then f is $C^{k+1} \Leftrightarrow Tf$ is C^k.*

Proof. Since this is a local theorem, it suffices to assume that V is convex. Note that, since an exponential space is compactly generated, $L(Y, Z)$ is complete (see [6, Proposition 3.23]). Also, since closed and bounded subsets of a complete Schwartz space are compact, the topology on $L(Y, Z)$ is the compact-open topology. Finally, by [6, Proposition 4.14] we only need to show that if Tf is C^k, then f is C^{k+1}.

We proceed by induction. Assume Tf is C^1. Since the projection of $Z \times Z$ onto the second factor is linear, the map

$$g: V \times Y \to Z$$
$$(x, y) \to Df(x)(y)$$

is C^1, so that the partial derivative with respect to the first coordinate, $D_1 g: V \times Y \to L(Y, Z)$, exists and is continuous. Since the evaluation map $ev: L(Y, Z) \times Y \to Z$ is continuous on compact sets, and $V \times Y$ is a k-space, it follows tha

$$h: V \times Y \times Y \to Z$$
$$(x, y, w) \to D_1 g(x, y)(w)$$

is continuous. Note that, since $g(x, y_1 + y_2) = g(x, y_1) + g(x, y_2)$ for each $y_1, y_2 \in Y$, we have that $h(x, y_1 + y_2, w) = h(x, y_1, w) + h(x, y_2, w)$ for each $y_1, y_2, w \in Y$. Since linearity in the third factor is obvious, it follows that $h(x, \cdot, \cdot) \in L^2(Y, Z)$ for each $x \in V$. Further, since the topology on $L^2(Y, Z)$ is the compact-open topology, a standard result from the topology of function spaces (see, for instance, [7, Theorem 7.5]) tells us that we may regard h as a continuous map from V to $L^2(Y, Z)$.

Now by the Fundamental Theorem applied to g and $D_1 g$ we see that for each $x, y \in V$ and $w \in Y$,

$$Df(y)(w) - Df(x)(w) = \int_0^1 h(ty + (1 - t)x)(w, y - x)\, dt,$$

which implies that

$$Df(y) - Df(x) = \int_0^1 h(ty + (1 - t)x)(y - x)\, dt.$$

By [6, Lemma 3.28] we obtain that Df is C^1, so that f is C^2.

Now let $r \in N$ with $r \geqslant 2$, and assume that the theorem is true for $k < r$. Let f be a C^1 map such that Tf is C^r. By the preceding part of the proof we find that f is C^2 so that Df is C^1. Now, $(T(Df))(x, y)(w) = (Df(x)(w), D^2f(x)(y, w))$, and $D_1(Tf)(x, y)(w) = (Df(x)(w), D^2f(x)(w, y))$ which implies that $T(Df) = D_1(Tf)$. Since Tf is C^r, $D_1(Tf)$ is C^{r-1}, i.e., $T(Df)$ is C^{r-1}. Applying the theorem for $(r - 1)$ to Df and $T(Df)$, we conclude that Df is C^r, so that f is C^{r+1}. Finally, applying induction, we see that the theorem is true for all $k \in N$. q.e.d.

The following corollary is the same as [6, Lemma 6.3]. The proof of that lemma, however, was not quite complete as it was presented in [6]. The proof does becomes correct if we are allowed to assume that the above theorem is valid. Thus the reader may refer to [6, p. 103] for the proof of the following.

5.3. Corollary. *Let X_1 and X_2 be bw* spaces, V be open in X_1, f: $V \to X_2$ be a $\mathfrak{U}^1$ map, and $k \in N$. Then f is $\mathfrak{U}^{k+1} \Leftrightarrow Tf$ is $\mathfrak{U}^k$.*

Now we return to the notation of Definition 4.1 and Theorem 4.2.

5.4. Theorem. *Let W be open in X, $k \in N$, f: $W \to X$ be a continuous vector field, and F: $V \to X$ be a flow generated by f.*

(a) *If f is C^k, then F is C^k.*

(b) *If f is $\mathfrak{U}^k$, then F is $\mathfrak{U}^k$.*

Proof. We proceed by induction. We know the theorem for $k = 1$ by Theorem 4.2 and Corollary 4.3. So let $k > 1$ and assume the theorem proved for the case $(k - 1)$.

Since F is C^1 in either case, we know that D_2F: $V \to L(X, X)$ exists and is continuous. Thus the map G: $V \times X \to X \times X$ defined by $G(t, x, y) = (F(t, x), D_2F(t, x)(y))$ is well-defined and continuous. Now recall that we showed in the proof of Theorem 4.2 that $(d/dt)(D_2F(t, x))$ exists and equals $Df(F(t, x))D_2F(t, x)$ for each $(t, x) \in V$. Thus for each $(t, x, y) \in V \times X$, $(d/dt)(G(t, x, y))$ exists and equals $(f(F(t, x)), Df(F(t, x))D_2F(t, x)(y))$. But

$$\begin{aligned}(f(F(t, x)), Df(F(t, x))D_2F(t, x)(y)) &= Tf(F(t, x), D_2F(t, x)(y)) \\ &= Tf(G(t, x, y)),\end{aligned}$$

i.e., $(d/dt)(G(t, x, y)) = Tf(G(t, x, y))$. Thus we have shown that G is a continuous flow for Tf. Since Tf has one degree of differentiability less than f, we may apply our inductive assumption to see that G is C^{k-1} if (a) holds, and that G is $\mathfrak{U}^{k-1}$ if (b) holds. Finally, note that we have the following expression for TF: $V \times \mathbf{R} \times X \to X \times X$:

$$\begin{aligned}TF(t, x)(s, y) &= (F(t, x), DF(t, x)(s, y)) \\ &= (F(t, x), D_1F(t, x)(s) + D_2F(t, x)(y)) \\ &= (F(t, x), sf(F(t, x)) + D_2F(t, x)(y)) \\ &= G((t, x, y)) + s(0, f(F(t, x))).\end{aligned}$$

Thus we conclude that TF is C^{k-1} if (a) holds, and that TF is $\mathfrak{U}^{k-1}$ if (b) holds. Hence applications of Theorem 5.2 and Corollary 5.3 complete the proof.

6. The existence of continuous flows

Since a C^∞ vector field is automatically strongly C^∞, the classical existence theorem on solutions to differential equations in Banach spaces implies that a weakly C^∞ vector field must generate a strongly C^∞ flow. However, we will present an example to show that there are C^∞ vector fields which do not generate flows in the (more restrictive) weak sense.

The example which follows was noticed independently by the author and J. P. Penot. This vector field, which is defined on the Hilbert space l_2 of square-summable sequences, has the property that the domain of the maximal flow generated by the vector field is not weakly open in $\mathbf{R} \times l_2$.

The other goal of this section is to show that a $\mathfrak{U}^1$ vector field does generate a (weakly) continuous flow.

6.1. Example. Define $f: l_2 \to l_2$ by $f((x_1, x_2, x_3, \cdots)) = (x_1^2, x_2^2, x_3^2, \cdots)$. Then f is obviously strongly C^1. To see that f is (weakly) C^1, it suffices to verify that Df is weakly continuous. But, since continuous linear maps are C^∞, once we have shown that Df is continuous, it will follow automatically that f is C^∞.

Let $\{e_i\}_{i \in N}$ be the canonical orthonormal basis for l_2, and regard each e_i as a linear functional on l_2 in the standard way. To see that Df is weakly continuous, we employ the technique of [6, Example 5.6]; namely, we show that $ev \circ (Df \times \mathrm{Id}): l_2 \times l_2 \to l_2$ is continuous. To see this, it suffices to show that Df maps strongly bounded sets to strongly bounded sets (refer to the just-cited example) and then to show that $e_i \circ ev \circ (Df \times \mathrm{Id}): l_2 \times l_2 \to \mathbf{R}$ is continuous for each $i \in N$.

Since Df is strongly continuous and linear, we know that Df maps strongly bounded sets to strongly bounded sets. To see that $e_i \circ ev \circ (Df \times \mathrm{Id})$ is continuous, note that $e_i \circ ev \circ (Df \times \mathrm{Id}) = ev \circ (Df_i \times \mathrm{Id})$, where f_i is the ith coordinate function of f. But $f_i: l_2 \to \mathbf{R}$ can be factored as $f_i = g \circ e_i$, where $g: \mathbf{R} \to \mathbf{R}$, $g(t) = t^2$. Thus each f_i is C^∞, which implies that Df_i is continuous, and hence that $ev \circ (Df_i \times \mathrm{Id})$ is continuous.

Let α_i be the maximal integral curve for f with initial value e_i. Then $\alpha_i(t) = e_i/(1 - t)$, and the domain of α_i is $(-\infty, 1)$. Thus if V is the domain of the maximal flow generated by f, then $(1, e_i) \notin V$ for $i \in N$. But, since $f(0) = 0$, $\mathbf{R} \times \{0\} \subset V$, and since $\lim_{i \to \infty}(1, e_i) = (1, 0)$, it follows that V is not weakly open in $\mathbf{R} \times l_2$. q.e.d.

The above example may be modified to produce a C^∞ vector field on l_2 which has a maximal flow whose domain is all of $\mathbf{R} \times l_2$ but which maps sufficiently large bounded subsets of $\mathbf{R} \times l_2$ to unbounded subsets of l_2. Since boundedness is equivalent to having compact closure in the weak topology, a flow which does not preserve boundedness cannot be weakly continuous. Thus a C^∞ vector field may generate a maximal flow with a weakly open domain such that the flow is not weakly continuous on that domain.

So we see that there are at least two reasons why a vector field might not generate a flow in our strengthened sense: the domain of the maximal flow might not be weakly open, and even if it is, the flow might map some weakly compact sets to unbounded sets.

The surprising thing is that these are essentially the only two things which can prevent a strong flow from being a flow. This is shown by the following lemma, which is due to S. Gautier and J. P. Penot [5].

6.2. Lemma. *Let $f: X \to X$ be a continuous vector field which has unique integral curves, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in X with limit x_0, and let $a < 0 < b$. For each $\lambda \in \Lambda$, assume that the domain of the maximal integral curve for f with initial value x_λ contains $[a, b]$, and denote the restriction of this curve to $[a, b]$ by α_λ. Assume that $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ is a bounded subset of $C^0([a, b], E)$. Then $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ is a convergent net in $C^0([a, b], X)$. Furthermore, if we denote the limit of this net by α_0, then α_0 is an integral curve for f with initial value x_0.*

Proof. Choose $n \in N$ such that $\alpha_\lambda([a, b]) \subset B_0(n)$ for each $n \in N$. Since $B_0(n)$ is compact in X, there exists $m \in N$ such that $\|f(x)\| \leqslant m$ for every $x \in B_0(n)$. Since $\alpha_\lambda(t) - \alpha_\lambda(s) = \int_s^t f(\alpha_\lambda(r))\, dr$ for each $t, s \in [a, b]$, it follows that $\|\alpha_\lambda(t) - \alpha_\lambda(s)\| \leqslant m|t - s|$. Thus $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ is an equicontinuous family of curves in $C^0([a, b], E)$, and since the identity map $i: E \to X$ is uniformly continuous, it follows that $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ is an equicontinuous family of curves in $C^0([a, b], X)$. But since $A = \bigcup_{\lambda \in \Lambda} \alpha_\lambda([a, b]) \subset B_0(n)$, A has compact closure in X, and hence by Ascoli's Theorem (see [7, p. 233]), $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ has compact closure in $C^0([a, b], X)$. It follows that the net has at least one limit point in $C^0([a, b], X)$.

Now let $\alpha_0 \in C^0([a, b], X)$ be any limit point of this net in $C^0([a, b], X)$, and let H be a subset of Λ such that $\lim_{\eta \in H} \alpha_\eta = \alpha_0$. Then for each $t, s \in [a, b]$ the uniform convergence of $\{\alpha_\eta\}_{\eta \in H}$ to α_0 implies that

$$\begin{aligned}\alpha_0(t) - \alpha_0(s) &= \lim_{\eta \in H} (\alpha_\eta(t) - \alpha_\eta(s)) \\ &= \lim_{\eta \in H} \int_s^t f(\alpha_\eta(r))\, dr = \int_s^t f(\alpha_0(r))\, dr,\end{aligned}$$

i.e., α_0 is an integral curve for f. Since $\alpha_0(0) = \lim_{\eta \in H} \alpha_\eta(0) = \lim_{\lambda \in \Lambda} \alpha_\lambda(0) = \lim_{\lambda \in \Lambda} x_\lambda = x_0$ and there is only one integral curve for f with initial value x_0, it follows that $[a, b]$ is contained in the domain of the maximal integral curve for f with initial value x_0, and that α_0 is the restriction of this curve to $[a, b]$. We also see that the net has only one limit point. But in a compact space, a net with only one limit point must be convergent. Thus $\lim_{\lambda \in \Lambda} \alpha_\lambda = \alpha_0$.

6.3. Theorem. *Let $k \in N \cup \{\infty\}$, and let $f: W \to X$ be a $\mathfrak{U}^k$ vector field. Then f generates a $\mathfrak{U}^k$ flow.*

Proof. It suffices to show that for each $x \in W$ there exists an open neighborhood W_x of x such that $f|_{W_x}$ generates a $\mathfrak{U}^k$ flow. So let $x \in X$. Then it follows easily from [6, Lemma 4.18] that there exists a C^∞ real-valued function $h: X \to [0, 1]$ such that support$(h) \subset W$ and such that there is a neighborhood V_x of x with $h|_{V_x} \equiv 1$. By [6, Corollary 6.12], a C^∞ map with finite-dimensional range is $\mathfrak{U}^\infty$, which implies that hf is $\mathfrak{U}^\infty$ on X, and that $(hf)|_{V_x} = f|_{V_x}$. Thus to show that $f|_{V_x}$ generates a flow it suffices to show that hf generates a $\mathfrak{U}^k$ flow, and hence to prove this theorem for the case $W = X$.

So assume that f: $X \to X$ is $\mathfrak{U}^k$, and let F: $V \to E$ be the maximal strong flow generated by f. To show that F is a $\mathfrak{U}^k$ map it suffices by Theorem 5.4 to show that V is open in $\mathbf{R} \times X$ and that F: $V \to X$ is weakly continuous.

Recall that $\mathbf{R} \times X$ is compactly generated. Thus to show that V is open in $\mathbf{R} \times X$ it suffices to show that $V_n = V \cap (\mathbf{R} \times (nK))$ is open in $\mathbf{R} \times (nK)$ for each $n \in N$.

So let $n \in N$, $x \in nK$, and let α: $(a, b) \to X$ be the maximal integral curve for f with initial value x. Let $\tau = \sup\{r \geqslant 0$: there exists a closed neighborhood A of x in nK such that $[0, r] \times A \subset V_n$ and $F|_{[0, r] \times A}$ is continuous$\}$, and let $\sigma = \inf\{r \leqslant 0$: there exists a closed neighborhood A of x in nK such that $[r, 0] \times A \subset V_n$ and $F|_{[r, 0] \times A}$ is continuous$\}$. To show that V_n is open in $\mathbf{R} \times (nK)$, it suffices to show that $\tau = b$ and $\sigma = a$. Note also that, once we have shown this, we will also have shown that F is continuous on V_n, since we will have shown that for each $t \in (a, b)$, there is a neighborhood of (t, x) in V_n on which F is defined and continuous. From this, the continuity of F on V is immediate.

To see that $\tau = b$ we first show that $\tau > 0$. Since f is continuous and hence bounded on weakly compact sets, there exists $m_1 \in N$ such that $\|f(z)\| \leqslant m_1$ for each $z \in (n + 1)K$. Letting $c = 1/m_1$, it follows that, for each $y \in nK$, the integral curve α_y for f with initial value y is defined on $[0, c]$, and that $\alpha_y([0, c]) \subset (n + 1)K$. By Lemma 6.2, it follows that F is continuous on $[0, c] \times (nK)$, and hence that $\tau \geqslant c$.

Now assume for the moment that, to the contrary, $\tau < b$. Then $\alpha([0, \tau])$ is compact in X, so there exists $m_2 \in N$ with $\|\alpha(t)\| \leqslant m_2$ for each $t \in [0, \tau]$. Choose $m_3 \in N$ such that $\|Df(\alpha(t))\| \leqslant m_3$ for each $t \in [0, \tau]$. Choose $m_4 \in N$ such that $m_4 \geqslant m_2 + 2ne^{(m_3+1)\tau}$. Choose m_5 such that $\|f(z)\| \leqslant m_5$ for each $z \in (m_4 + 1)K$. Finally, let $\varepsilon = 1/2m_5$. It follows as in the preceding paragraph that for each $y \in m_4K$ the integral curve α_y for f with initial value y is defined on $[0, 2\varepsilon]$ and that $\alpha_y([0, 2\varepsilon]) \subset (m_4 + 1)K$.

So choose a closed neighborhood C of x in nK such that $[0, \tau - \varepsilon] \times C \subset V_n$ and that $F|_{[0, \tau-\varepsilon] \times C}$ is continuous. Since $[0, \tau - \varepsilon] \times C$ is compact, $F([0, \tau - \varepsilon] \times C)$ is compact. Hence the closed convex hull of $F([0, \tau - \varepsilon] \times C)$ in X, denoted by D, is also compact. Since f is $\mathfrak{U}^1$, there exists a continuous semi-norm λ on X such that, for each $y, z \in D$ and $w \in X$,

$$\|(Df(z) - Df(y))w\| \leqslant \lambda(z - y)\|w\| + \|z - y\|\lambda(w).$$

By continuity, there exists a closed neighborhood W of x in C such that $\lambda(F(t, y) - F(t, x)) \leqslant 1$ for all $y \in W$ and all $t \in [0, \tau - \varepsilon]$. Note that W is automatically a closed neighborhood of x in V_n. Furthermore, for each $y \in W$ and $t \in [0, \tau - \varepsilon]$,

$$\begin{aligned} F(t,y) - F(t,x) &= (y-x) + \int_0^t (f(F(s,y)) - f(F(s,x)))\,ds \\ &= (y-x) \\ &\quad + \int_0^t \int_0^1 Df(rF(s,y) + (1-r)F(s,x))(F(s,y) - F(s,x))\,drds \\ &= (y-x) + \int_0^t Df(F(s,x))(F(s,y) - F(s,x))\,ds \\ &\qquad + \int_0^t \int_0^1 (Df(rF(s,y) + (1-r)F(s,x)) - Df(F(s,x))) \\ &\qquad\qquad \cdot (F(s,y) - F(s,x))\,drds, \end{aligned}$$

which implies that

$$\begin{aligned} &\|F(t,y) - F(t,x)\| \\ &\quad \leqslant \|y-x\| + m_3 \int_0^t \|F(s,y) - F(s,x)\|\,ds \\ &\qquad + \int_0^t \int_0^1 2r\|F(s,y) - F(s,x)\|\lambda(F(s,y) - F(s,x))\,drds \\ &\quad = \|y-x\| + m_3 \int_0^t \|F(s,y) - F(s,x)\|\,ds \\ &\qquad + \int_0^t \|F(s,y) - F(s,x)\|\lambda(F(s,y) - F(s,x))\,ds \\ &\quad \leqslant \|y-x\| + (m_3+1)\int_0^t \|F(s,y) - F(s,x)\|\,ds. \end{aligned}$$

By Gronwall's inequality,

$$\|F(t,y) - F(t,x)\| \leqslant \|y-x\|e^{(m_3+1)t} \leqslant 2ne^{(m_3+1)\tau},$$

and in particular, we have

$$\|F(\tau - \varepsilon, y)\| \leqslant m_2 + 2ne^{(m_3+1)\tau} \leqslant m_4$$

for each $y \in W$. Thus the integral curve for f with initial value $F(\tau - \varepsilon, y)$ is defined on $[0, 2\varepsilon]$ and is contained in $(m_4 + 1)K$, which means that $[0, \tau + \varepsilon] \times W \subset V_n$ and $\|F(t,y)\| \leqslant m_4 + 1$ for each $t \in [0, \tau + \varepsilon], y \in W$. An application of Lemma 6.2 tells us that F is continuous on $[0, \tau + \varepsilon] \times W$, which contradicts the maximality of τ. Thus τ must equal b as was originally asserted.

The proof that $\sigma = a$ is essentially identical to the proof that $\tau = b$, and so is omitted.

7. Manifolds of maps

We will assume throughout this section that M is a compact n-dimensional manifold for some positive integer n. Let N be a $\mathcal{U}^\infty$ Bw^* manifold (for the definition and basic properties of these objects, see [6, Chapter 7]). Then TN, the tangent bundle of N, is a $\mathcal{U}^\infty$ manifold, and it is possible to use a $\mathcal{U}^\infty$ partition of unity on N to construct a $\mathcal{U}^\infty$ spray on TN. Thus if $\mathfrak{M}$ is a section functor (for instance, the Lipschitz functor Lip_k, the Hölder functor $C^{k,\varepsilon}$, or the Sobolev functor L_k^p with $k > n/p$), the question naturally arises whether $\mathfrak{M}(M, N)$, the space of maps from M to N of class $\mathfrak{M}$, has the structure of a $\mathcal{U}^\infty$ Bw^* manifold in the obvious naturally-arising way. The answer to this question is: almost, but not quite. We will spend the rest of this section discussing this answer in some detail.

Let X be a Bw^* space, $E = X''$, and D^n be the n-dimensional unit disc. Then it is easy to see, using the results of [6, 2.1–2.21], that $\mathrm{Lip}_k(D^n, E)$ and $C^{k,\varepsilon}(D^n, E)$ have a Bw^* topology which is induced by the inclusion of these spaces in $C^0(D^n, X)$ ($k \in N \cup \{0\}$, $\varepsilon \in (0, 1)$), and it follows from [4, §8.20] that $L_k^p(D^n, E)$ (appropriately defined) is a Bw^* space for $p > 1$ and $k > n/p$. These Bw^* spaces will be denoted by $\mathrm{Lip}_k(D^n, X)$, $C^{k,\varepsilon}(D^n, X)$, and $L_k^p(D^n, X)$ respectively.

7.1 Definition. Let (ξ, π) be a pair such that ξ is a $\mathcal{U}^\infty$ manifold, and $\pi: \xi \to M$ a $\mathcal{U}^\infty$ map. Assume that, for each $x \in M$, $\pi^{-1}(x)$ has the structure of a Bw^* space. The pair (ξ, π) will be called a $\mathcal{U}^\infty$ vector bundle over M if for each $x \in M$ there exist a Bw^* space X, an open neighborhood U of x in M, and a $\mathcal{U}^\infty$ fiber-preserving diffeomorphism $\phi: \pi^{-1}(U) \to U \times X$, where $U \times X$ is regarded as a bundle over M by projection onto the first factor, such that $p_2 \circ \phi: \pi^{-1}(y) \to X$ is a linear isomorphism of Bw^* spaces for each $y \in U$.

From the discussion preceding Definition 7.1 it is obvious that the Lip_k-sections of ξ form a Bw^* space as do the $C^{k,\varepsilon}$-sections and the L_k^p-sections. The methods developed in §§1 and 6 of [6] for finite-dimensional bundles over M now show (without modification) that Lip_k, $C^{k,\varepsilon}$ and L_k^p are functors from the category of $\mathcal{U}^\infty$ vector bundles over M and $\mathcal{U}^\infty$ fiber-preserving maps between them to the category of Bw^* spaces and C^∞ maps. Unfortunately, as the following example shows, the induced maps are not in general of differentiability class $\mathcal{U}^\infty$, though they are for $k \geqslant 1$ if the bundles are finite-dimensional, as is shown in [6, Chapter 6].

7.2. Example. Let $H_1(S^1, \mathbf{R})$ be the Hilbert space of absolutely continuous functions on S^1 with square-integrable derivatives, and let X be $H_1(S^1, \mathbf{R})$ with the bounded-weak topology. Then the map $f: \underset{r \to r^2}{\mathbf{R} \to \mathbf{R}}$ induces a $\mathcal{U}^\infty$

map $H_1(f)\colon X \to X$ (in the terminology of the preceding paragraphs, H_1 is the Sobolev functor L_1^2). By the above remarks, the $\mathcal{U}^\infty$ maps $H_1(f)$ in turn induces a weakly C^∞ map $H_1(H_1(f))\colon H_1(S^1, X) \to H_1(S^1, X)$. However, the map $H_1(H_1(f))$ is not $\mathcal{U}^1$.

Proof. Let $T^2 = S^1 \times S^1$, and define $H_{1,1}(T^2, \mathbf{R})$ to be the Hilbert space of square-integrable functions whose first distributional partial derivatives and second *mixed* partial derivatives are square-integrable. For each $s \in H_1(S^1, H_1(S^1, \mathbf{R}))$, we get a function $\phi(s) \in H_{1,1}(T^2, \mathbf{R})$ defined by $(\phi(s))(t, r) = (s(t))(r)$. It is well-known (see [3]) that ϕ is a linear isomorphism. Under this isomorphism, $H_1(H_1(f))$ corresponds to the map $H_{1,1}(f)\colon H_{1,1}(T^2, \mathbf{R}) \to H_{1,1}(T^2, \mathbf{R})$ defined by $H_{1,1}(f)(s) = s^2$. So it suffices to show that $H_{1,1}(f)$ is not $\mathcal{U}^1$ on $H_{1,1}(T^2, \mathbf{R})$.

To simplify the notation in what follows, denote $H_{1,1}(f)$ by g. Note that

$$\left\{ \frac{e^{irt}e^{imr}}{2\pi\sqrt{1 + r^2 + m^2 + r^2m^2}} \right\}_{n,m\in\mathbf{Z}}$$

is an orthonormal basis for $H_{1,1}(T^2, \mathbf{R})$. Note also that $Dg(0) = 0$. Define two sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$, both of which converge weakly to zero, by

$$x_n = \frac{e^{int}}{2\pi\sqrt{1 + n^2}}, \quad y_n = \frac{e^{inr}}{2\pi\sqrt{1 + n^2}}.$$

If g were of differentiability class $\mathcal{U}^1$ then, since $\{0\} \cup \{x_n\}_{n\in N}$ is weakly compact, it would follow that

$$\lim_{n\to\infty} \|Dg(x_n)(y_n)\| = \lim_{n\to\infty} \|(Dg(x_n) - Dg(0))y_n\| = 0.$$

However, by direct computation, it follows that

$$Dg(x_n)(y_n) = \frac{e^{int}e^{inr}}{2\pi^2(1 + n^2)},$$

which implies that $\|Dg(x_n)(y_n)\| = 1/\pi^2$ for every $n \in N$. Thus g cannot be $\mathcal{U}^1$. q.e.d.

Since we have the inverse function theorem for $\mathcal{U}^\infty$ maps, and we know that $\mathcal{U}^\infty$ vector fields generate $\mathcal{U}^\infty$ flows, we can establish a generalization of Palais' vector bundle neighborhood theorem for finite-dimensional fiber bundles [9, Theorem 12.10]; namely, if G is a $\mathcal{U}^\infty$ fiber bundle over M, $s \in C^0(G)$, then there exist a $\mathcal{U}^\infty$ vector bundle ξ over M, an open neighborhood V of the zero section in ξ, an open neighborhood W of $s(M)$ in G, and a $\mathcal{U}^\infty$ fiber-preserving diffeomorphism $f\colon V \to W$. In fact, Palais' original proof may be used in this more general setting.

With the generalized vector bundle neighborhood theorem available, it is easy to redo the constructions of Chapters 1 and 6 of [6] to show that any section functor from the category of $\mathcal{U}^\infty$ vector bundles over M and $\mathcal{U}^\infty$ fiber-preserving maps between them to the category of Bw^* spaces and continuous maps extends in exactly one way to a functor from the category of $\mathcal{U}^\infty$ fiber bundles over M and $\mathcal{U}^\infty$ fiber-preserving maps between them to the category of C^∞ Bw^* manifolds and weakly C^∞ maps. However, as Example 7.2 shows, if G is a $\mathcal{U}^\infty$ fiber bundle over M, then we should not expect the resulting Bw^* manifold to be $\mathcal{U}^\infty$ if the bundle is not finite-dimensional.

8. Remarks

The classical proof of the Banach space theorem on the existence of a differentiable flow for a differentiable vector field consists of the three distinct parts: construction of a continuous (indeed, locally uniformly Lipschitz) flow, a separate proof that any continuous flow must be continuously differentiable, and some sort of induction to show that the flow possesses as high a degree of differentiability as the vector field. The same three distinct parts occur in the proof that a $\mathcal{U}^k$ vector field generates a $\mathcal{U}^k$ flow. Since the results on differentiability of the flow apply to more vector fields on bw^* spaces than just the $\mathcal{U}^k$ vector fields, the author has chosen to put the proof of differentiability in front of the proof of the existence of a weakly continuous flow; but the order in which these parts are proved does not matter, since they do not depend on each other.

A comparison of the proof of Theorem 4.2 with that of differentiability of the flow in the Banach space case is in order. Of course, the standard proof of differentiability in the Banach space case depends on the solution of the appropriate time-dependent linear equation. However, proofs in most texts are somewhat more complicated than the proof of Theorem 4.2 because they fail to make full use of the evolution system generated by the linear equation (specifically, of the partial derivative with respect to the second variable, $(\partial/\partial s)(U(t, s))$). Note that the analogues of the results of §3 are essential to the technique of proof used in Theorem 4.2: continuously parametrized time-dependent linear equations of the type under consideration must a priori be known to generate evolution systems which depend continuously on the parameter space.

This technique can be used to show differentiability of a continuous flow in other situations than the classical case and the one considered in Theorem 4.2. For instance, let E be a Banach space, W be open in E, $[a, b] \subset \mathbf{R}$, and $g: W \times [a, b] \to L(E, E)$ be a parameterized family of operators which is continuous from $W \times [a, b]$ to $L(E, E)$ with the strong operator topology.

Then $g(\cdot,\cdot)$ generates a unique parametrized evolution system $U(\cdot,\cdot,\cdot)$: $W \times [a, b] \times [a, b] \to L(E, E)$, and it is easy to see that $U(\cdot,\cdot,\cdot)$ is continuous from its domain to $L(E, E)$ with the strong operator topology. Thus the proof of Theorem 4.2 adapts to the present setting with no essential modification to yield the following.

8.1. Theorem. *Let E be a Banach space, W open in E, f: $W \to E$ a vector field which has a Gateaux derivative at each point of W. Assume that Df: $W \to L(E, E)$ is continuous from W to $L(E, E)$ with the strong operator topology. Then f generates a flow F with domain V open in $\mathbf{R} \times E$ such that F has a Gateaux derivative at each point of V and that DF: $V \to L(\mathbf{R} \times E, E)$ is continuous from V to $L(\mathbf{R} \times E, E)$ with the strong operator topology.*

Proof. The assumption that Df is strongly continuous implies that f is locally uniformly Lipschitz, and hence that f generates a continuous flow F. The rest now follows from the paragraph preceding this theorem using the technique of proof of 4.2. q.e.d.

This approach to the differentiability of a continuous flow can also be applied to flows and semi-flows generated by nonlinear partial differential operators, different cases of which were discovered independently and more or less simultaneously by J. R. Dorroh and J. E. Marsden. In fact, the technique in the proof of Theorem 4.2 has its origin in the work of Marsden on this application [8, pp. 279–284]. Marsden's treatment applies to the case where parametrized families of unbounded linear operators of the type under consideration generate parametrized evolution systems which are known a priori to depend continuously upon the parameter. A modified version of Dorroh's method (developed by the present author–see [2]) applies to a situation where this continuous dependence is not assumed a priori but is demonstrated during the course of the proof of differentiability. It turns out that, in this case, it is only necessary to assume that families of unbounded linear operators of appropriate type generate what Dorroh and the author have called a weak evolution system (which exists under somewhat more natural conditions from the point of view of applications than an evolution system).

The author has developed linear evolution systems on bw^* spaces in §3 of this paper. The reader who is unfamiliar with the concept of a linear evolution system and wishes to compare §3 with a treatment in a more conventional setting may wish to consult [1] for a particularly readable treatment of evolution systems generated by families of bounded linear operators on a Banach space, or [2] for a new and thorough treatment of evolution systems generated by families of unbounded linear operators in a Banach space.

Incidentally, note that the results of this paper apply to two opposing extreme cases: on the one hand, we have our main result, Theorem 6.3, which represents a definite strengthening of the corresponding result in the category of Banach spaces and differentiable maps; and on the other hand, we have Theorems 4.2 and 8.1, which tell us that a flow in a Banach space possesses some sort of continuous differentiability in certain instances where the generating vector fields possesses a degree of differentiability which is less than C^1 in the classical sense of Fréchet differentiability in Banach spaces.

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LOUISIANA STATE UNIVERSITY